

STANLEY DEPTH AND THE LCM-LATTICE

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ABSTRACT. In this paper we show that the Stanley depth, as well as the usual depth, are essentially determined by the lcm-lattice. More precisely, we show that for quotients I/J of monomial ideals $J \subset I$, both invariants behave monotonic with respect to certain maps defined on their lcm-lattice. This allows simple and uniform proofs of many new and known results on the Stanley depth. In particular, we obtain a generalization of our result on polarization presented in [IKMF14]. We also obtain a useful description of the class of all monomial ideals with a given lcm-lattice, which is independent from our applications to the Stanley depth.

1. INTRODUCTION

Let \mathbb{K} be a field, S a \mathbb{N}^n -graded \mathbb{K} -algebra and M a finitely generated \mathbb{Z}^n -graded S -module. The *Stanley depth* of M , denoted $\text{sdepth}_S M$, is a combinatorial invariant of M related to a conjecture of Stanley from 1982 [Sta82, Conjecture 5.1] (nowadays called the *Stanley conjecture*), which states that $\text{depth}_S M \leq \text{sdepth}_S M$. Starting with the work of Apel [Ape03a, Ape03b], a lot of research has been devoted to the study of this conjecture. We refer the reader to the survey by Herzog [Her13] for a comprehensive account of the known results. Most of the research concentrates on the particular case of a module of the form I/J for two monomial ideals $J \subsetneq I$ in the polynomial ring $S = \mathbb{K}[X_1, \dots, X_n]$. In the present paper we will also work in this setting.

Many known results on the Stanley depth are paralleled by older results on (the usual) depth. For example, both invariant are bounded above by the dimension of the associated primes of I/J . The aim of the current paper is to understand the reason behind several of these phenomena. To this end, we study the *lcm-lattice* of the ideals. For the sake of simplicity, we further restrict this presentation to the case of modules of the form S/I , even though most of our results are proven more generally.

The lcm-lattice L_I of $I \subset S$ is the lattice of all least common multiples of subsets of the (minimal) generators of I , ordered by divisibility, see Gasharov et al. [GPW99]. It is a finite atomistic lattice that is known to encode a lot of information about I . In particular, it encodes the structure of the minimal free resolution of S/I over I and thus determines the Betti numbers and projective dimension of S/I [GPW99, Theorem 3.3]. More precisely,

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what is shown in [GPW99] is the following: Let $I \subset S$ and $I' \subset S'$ be two monomial ideals. Then, given a free resolution of S/I and a surjective join-preserving map $L_I \rightarrow L_{I'}$, one can construct a free resolution of S'/I' by a certain relabeling procedure. In particular, the projective dimension of S'/I' is bounded above by the projective dimension of S/I .

The main result we obtain implies in particular the analogous statement for the Stanley depth.

Theorem (Theorem 4.5). *Let $I \subset S$ and $I' \subset S'$ be two monomial ideals in two polynomial rings in n resp. n' variables. If there exists a surjective join-preserving map $L_I \rightarrow L_{I'}$, then*

$$n - \text{sdepth}_S S/I \geq n' - \text{sdepth}_{S'} S'/I'.$$

(For technical reasons, the actual Theorem 4.5 is formulated in terms of lcm-semilattices, i.e. lcm-lattices without minimal elements. See Remark 2.3 for a discussion.) In view of the foregoing formula, we introduce the *Stanley projective dimension* as $\text{spdim}_S S/I := n - \text{sdepth}_S S/I$.

This result has a number of striking consequences. First of all, it shows that two ideals with isomorphic lcm-lattices have the same Stanley projective dimension. Thus, this invariant is determined by the isomorphism type of the lcm-lattice. However, unlike for the usual projective dimension, we do not know a direct way to read off the Stanley projective dimension from the lattice (similar to [GPW99]). As a special case, the lcm-lattice of an ideal is invariant under polarization. Hence Theorem 4.5 generalizes the main result of [IKMF14], where we showed that the Stanley projective dimension is invariant under polarization. However, we use this result in the proof of Theorem 4.5, thus the result on polarization is not simply a corollary.

Next, we present a simple and uniform proof for upper bounds on the Stanley projective dimension (i.e. lower bounds on the Stanley depth) in terms of the number of generators in Proposition 5.2. This just follows from the fact that every atomistic lattice on k atoms is the image of a boolean lattice on k generators. While these bounds are known (see Shen [She09], Keller and Young [KY09], and Okazaki [Oka11]), one may see this proof as simpler and more conceptual than the original proofs. We also characterize the extremal case and prove the Stanley conjecture for ideals with $\text{pdim}_S S/I = k - 1$, where k is the number of generators of I . As another application we study generic deformations in Proposition 5.5 and the forming of colon ideals in Proposition 5.8. The latter allows to give in Corollary 5.9 a uniform proof that both depth and Stanley depth are bounded by the dimensions of the associated prime ideals. Moreover, in Proposition 5.12 we show that for proving the Stanley conjecture one may always assume that the ideal under consideration is generated in a single degree.

We further identify some operations on ideals, e.g. passing to the radical, that yield surjective join-preserving maps on the lcm-lattice, so we obtain inequalities for the Stanley projective dimension in these cases. As all our proofs rest on Theorem 4.5 and we showcase an analogous result for the usual projective dimension (Theorem 4.9), we obtain the same bounds as for the usual projective dimension. While these results are well-known, it is relevant that we obtain *uniform* proofs for both depth and Stanley depths, thus explaining the observed parallel behavior. Furthermore, we prove an interesting reduction of the

Stanley conjecture in Theorem 5.14: To show it, one may always assume that the ideal in question is a localization of an ideal which already satisfies the conjecture.

In the way of studying the relation of ideals to their lcm-lattices, we also get a result of independent interest. In Theorem 3.4 we give a complete description of the class of all monomial ideals with a given lcm-lattice. This result also allows the easy construction of monomial ideals with a prescribed lcm-lattice, which we consider very useful for the study of examples.

Finally, the fact that both the projective dimension and the Stanley projective dimension are determined by the lcm-lattice allows further to formulate the Stanley conjecture completely in terms of finite lattices. So one can try to apply notions and techniques from this field to the Stanley conjecture. In the last section we indicate some of these ideas. In particular cases, this allows to reduce the study of infinitely many monomial ideals to finitely many finite lattices. This paves the way to computations. In a computational experiment, we have classified all lcm-lattices of ideals I with up to 5 generators and found that the Stanley conjecture holds for both I and S/I ; this is work in progress and it remains to be presented in our forthcoming article [IKMF].

2. PRELIMINARIES

2.1. Finite lattices and semilattices. A *(join-)semilattice* L is a partially ordered set (L, \leq) such that, for any $P, Q \in L$, there is a unique least upper bound $P \vee Q$ called the join of P and Q . A *lattice* is a join-semilattice L with the additional property that for any $P, Q \in L$, there is a unique greatest lower bound $P \wedge Q$ called the meet of P and Q . One can also define meet-semilattices in the obvious way. However, in the present paper we will never consider meet-semilattices, hence in the sequel *the term semilattice always refers to a join-semilattice*.

We are mainly interested in *finite* semilattices. Every finite semilattice has a unique maximal element $\hat{1}$. Moreover, a finite semilattice is a lattice if and only if it has a minimal element. So we can associate to every finite semilattice L a canonical lattice $\bar{L} := L \cup \{\hat{0}\}$ by adjoining a minimal element $\hat{0}$. Unless stated otherwise, all semilattices in the sequel will be assumed to be finite and \bar{L} will always denote the associated lattice.

We say that an element $N \in L$ *covers* another element $M \in L$, if $M < N$ and there exists no other element $N' \in L$, such that $M < N' < N$. An element is called an *atom* if it covers the minimal element $\hat{0}$ in \bar{L} . Equivalently, the atoms are the minimal elements of L (in the sense that there are no smaller elements). We call L *atomistic*, if every element can be written as a join of atoms.

An element N of a semilattice L is called *join-irreducible* if $N = P \vee Q$ implies $N = P$ or $N = Q$ for $P, Q \in L$. A *meet-irreducible* element is an element which is covered by exactly one other element. This terminology is justified by noting that N is meet-irreducible if and only if $N = P \wedge Q$ implies $N = P$ or $N = Q$ for $P, Q \in \bar{L}$ where the meet is taken in L . A *join-preserving map* $\phi : L \rightarrow L'$ is a map with $\phi(P \vee Q) = \phi(P) \vee \phi(Q)$ for all $P, Q \in L$. Remark that every join-preserving map preserves the order.

The next lemma collects several well-known facts about (finite) semilattices we will use in the sequel.

Lemma 2.1. *Let L be a (finite) semilattice. The following statements hold:*

- (1) Every non-maximal (resp. non-minimal) element $M \in L$ can be written as a meet (resp. join) of meet-irreducible (resp. join-irreducible) elements (where the meet is taken in \bar{L}).
- (2) For two elements $M, N \in L$, it holds that $M \leq N$ if and only if every meet-irreducible element greater than or equal to N is also greater than M .

Proof. The statement (1) can be found in Davey and Priestley [DP02, Chapter 2]. The statement (2) follow directly from (1). \square

2.2. The lcm-semilattice of a set of monomials. Let $S = \mathbb{K}[X_1, \dots, X_n]$ be a polynomial ring. A monomial $m \in S$ is a product of powers of variables of S . In particular, $1_{\mathbb{K}}$ is a monomial, but $0_{\mathbb{K}}$ is not.

Definition 2.2. The lcm-semilattice $L_G \subset S$ of a finite set $G \subset S$ of monomials is defined as the set of all monomials that can be obtained as the least common multiple (lcm) of some non-empty subset of G , ordered by divisibility.

Note that $G \subset L_G$. We will occasionally consider the associated lcm-lattice $\bar{L}_G = L_G \cup \{\hat{0}\}$. Note that $\hat{0}$ could be regarded as the lcm of the empty set, but we do not identify $\hat{0}$ with $1_{\mathbb{K}}$. For a monomial ideal $I \subset S$ with $I \neq (0)$ we define its lcm-semilattice as $L_I := L_{G(I)}$, where $G(I)$ is a minimal generating set of I . For the zero ideal we set $L_{(0)} := \emptyset$ and (consequently) $\bar{L}_{(0)} = \{\hat{0}\}$. Note that L_G is a finite join-semilattice, and that it is atomistic if and only if the elements of G form a minimal generating set of some monomial ideal (G). In this case, the atoms of L_G are exactly the elements of G .

Remark 2.3. The definition of the lcm-lattice \bar{L}_I of a monomial ideal given above coincides with the one given in [GPW99], except in the case $I = S$. In this case our lcm-lattice is $L_S = \{\hat{0}, 1_{\mathbb{K}}\}$, while the one of [GPW99] is just $\{1_{\mathbb{K}}\}$. While this difference is minor, we think that our definition is more convenient to work with. For example, the lcm-lattice of S should be isomorphic to the lcm-lattice of a principal ideal.

As the lcm-lattice contains the same information as the lcm-semilattice one might wonder why we introduce the latter. The main reason for this is again technical convenience. We are going to study maps $\phi : \bar{L} \rightarrow \bar{L}'$ of the lcm-(semi)lattices below and we usually require that the preimage of $\hat{0}$ is just $\hat{0}$. This condition is automatically satisfied for maps defined on the semilattices. Moreover, this saves us from the occasional need to exclude $\hat{0}$ from our considerations.

Finally, from a systematic point of view it seems reasonable to consider semilattices as we are also only considering maps in that category.

2.3. Stanley depth and polarization. Consider the polynomial ring S endowed with the multigraded structure. Let M be a finitely generated graded S -module, and let λ be a homogeneous element in M . Let $Z \subset \{X_1, \dots, X_n\}$ be a subset of the set of indeterminates of S . The $\mathbb{K}[Z]$ -submodule $\lambda \mathbb{K}[Z]$ of M is called a *Stanley space* of M if $\lambda \mathbb{K}[Z]$ is free (as $\mathbb{K}[Z]$ -submodule). A *Stanley decomposition* of M is a finite family

$$\mathcal{D} = (\mathbb{K}[Z_i], \lambda_i)_{i \in \mathcal{I}}$$

in which $Z_i \subset \{X_1, \dots, X_n\}$ and $\lambda_i \mathbb{K}[Z_i]$ is a Stanley space of M for each $i \in \mathcal{I}$ with

$$M = \bigoplus_{i \in \mathcal{I}} \lambda_i \mathbb{K}[Z_i]$$

as a multigraded \mathbb{K} -vector space. This direct sum carries the structure of S -module and has therefore a well-defined depth. The *Stanley depth* $\text{sdepth } M$ of M is defined to be the maximal depth of a Stanley decomposition of M . The Stanley conjecture states the inequality

$$\text{sdepth } M \geq \text{depth } M.$$

In the same fashion we introduce the following definition.

Definition 2.4. The *Stanley projective dimension* $\text{spdim}_S M$ of M is the minimal projective dimension of a Stanley decomposition of M .

Remark that $\text{spdim}_S M = n - \text{sdepth}_S M$ by the Auslander-Buchsbaum formula. While this definition might seem redundant, it turns out that our results (for example, see Theorem 4.5) are more naturally stated in terms of the Stanley projective dimension. Further, the Stanley conjecture may be reformulated in this context as follows:

Conjecture 2.5. $\text{pdim } M \geq \text{spdim } M$.

Recall that polarization is a deformation process assigning to an arbitrary monomial ideal a squarefree monomial ideal in a new set of variables, see e.g. Herzog and Hibi [HH11] for a precise definition. An easy corollary of [IKMF14, Theorem 4.3] is the following.

Theorem 2.6. Let $J \subsetneq I \subset S$ be monomial ideals, and let $I^p, J^p \subset S^p$ be their polarizations. Then $J^p \subsetneq I^p$ and

$$\text{spdim}_S I/J = \text{spdim}_{S^p} I^p/J^p.$$

In the proof of this result, certain type of poset maps plays a crucial role:

Definition 2.7. [IKMF14, Definition 3.1] Let $\ell \in \mathbb{Z}$ and $n, n' \in \mathbb{N}$. A monotonic map $\phi : \mathbb{N}^n \rightarrow \mathbb{N}^{n'}$ is said to *change the Stanley depth* by ℓ with respect to $g \in \mathbb{N}^n$ and $g' \in \mathbb{N}^{n'}$, if it satisfies the following two conditions:

- (1) $\phi(g) \leq g'$.
- (2) For each interval $[a', b'] \subset [0, g']$, the (restricted) preimage $\phi^{-1}([a', b']) \cap [0, g]$ can be written as a finite disjoint union $\bigcup_i [a^i, b^i]$ of intervals, such that

$$\#\{j \in [n] : b_j^i = g_j\} \geq \#\{j \in [n'] : b'_j = g'_j\} + \ell \quad \text{for all } i.$$

Those maps were profusely studied in [IKMF14]; we briefly recall two results.

Proposition 2.8. [IKMF14, Proposition 3.3] Let $n, n' \in \mathbb{N}$, $S = \mathbb{K}[X_1, \dots, X_n]$ and $S' = \mathbb{K}[X_1, \dots, X_{n'}]$ be two polynomial rings and let $J' \subsetneq I' \subset S'$ be monomial ideals. Consider a monotonic map $\phi : \mathbb{N}^n \rightarrow \mathbb{N}^{n'}$ and set $I := \Phi^{-1}(I')$, $J := \Phi^{-1}(J')$. Choose $g \in \mathbb{N}^n$ and $g' \in \mathbb{N}^{n'}$, such that every minimal generator of I and J divides X^g , and every minimal generator of I' and J' divides $X^{g'}$.

Let $\ell \in \mathbb{Z}$ and assume that ϕ changes the Stanley depth by ℓ with respect to g and g' . Then

- (i) I and J are monomial ideals, and
- (ii) $\text{sdepth}_S I/J \geq \text{sdepth}_{S'} I'/J' + \ell$.

Lemma 2.9. [IKMF14, Lemma 3.4] *Let $n_1, n'_1, n_2, n'_2 \in \mathbb{N}$. For $i = 1, 2$, let $\phi_i : \mathbb{N}^{n_i} \rightarrow \mathbb{N}^{n'_i}$ be monotonic maps that change the Stanley depth by ℓ_i with respect to $g_i \in \mathbb{N}^{n_i}$ and $g'_i \in \mathbb{N}^{n'_i}$. Then the product map*

$$(\phi_1, \phi_2) : \mathbb{N}^{n_1+n_2} \rightarrow \mathbb{N}^{n'_1+n'_2}$$

changes the Stanley depth by $\ell_1 + \ell_2$ with respect to (g_1, g_2) and (g'_1, g'_2) .

3. A STANDARD MAP ON THE LCM-SEMILATTICE

Let $S = \mathbb{K}[X_1, \dots, X_n]$ be a polynomial ring and let $G \subset S$ be a finite set of monomials. For a monomial $m \in S$, we define $\text{ord}_i(m)$ to be the exponent of X_i in m . We extend this definition to \overline{L}_G by setting $\text{ord}_i(\hat{0}) := 0$ for all i .

We define a *standard map* $w : \overline{L}_G \rightarrow S$ associating to each element $m \in \overline{L}_G$ a monomial $w_G(m)$ as follows: For the minimal and maximal elements $\hat{0}, \hat{1} \in \overline{L}_G$, we define $w_G(\hat{0}) := \gcd\{p \in G\}$ and $w_G(\hat{1}) := 1_{\mathbb{K}}$. For every other m we set

$$w_G(m) := \frac{1}{m} \gcd\{p \in L : p > m\}.$$

For w_G we deduce the following inversion formula:

Proposition 3.1. *For $m \in L_G$ it holds that*

$$m = \prod_{\substack{q \in \overline{L}_G \\ q \not\leq m}} w_G(q).$$

Proof. First we introduce some notations. For $1 \leq i \leq n$, consider the set

$$\mathcal{O}_i := \{o_{i,0} < o_{i,1} < o_{i,2} < \dots < o_{i,r_i} < o_{i,r_i+1}\} := \{\text{ord}_i(m) : m \in \overline{L}_G\}.$$

(Note that $0 = o_{i,0} = \text{ord}_i(\hat{0})$ for all i .) For each $1 \leq i \leq n$ such that $\mathcal{O}_i \neq \{0\}$ (i.e. $X_i \mid \hat{1}$) and $0 \leq j \leq r_i + 1$, let f_j^i be the join of all $m \in \overline{L}_G$ such that $\text{ord}_i(m) \leq o_{i,j}$. It is clear that $f_0^i < f_1^i < \dots < f_{r_i}^i < f_{r_i+1}^i$ and that $\text{ord}_i(f_j^i) = o_{i,j}$. Note that $f_{r_i+1}^i$ is the maximal element $\hat{1}$ of L_G for each i .

Observe that the elements $f_0^i, \dots, f_{r_i}^i$ are exactly those elements $m \in \overline{L}_G$ for which $X_i \mid w_G(m)$. Moreover, it follows from the definition of w_G that

$$\text{ord}_i(w_G(f_j^i)) = \text{ord}_i(f_{j+1}^i) - \text{ord}_i(f_j^i) = o_{i,j+1} - o_{i,j}$$

for $0 \leq j \leq r_i$. Let $m \in L_G$ be a monomial and let $1 \leq i \leq n$ such that $\mathcal{O}_i \neq \{0\}$. Then $\text{ord}_i(m) = o_{i,j}$ for some j . Therefore

$$\text{ord}_i(m) = \text{ord}_i(f_j^i) = \sum_{k=1}^{j-1} \text{ord}_i(w_G(f_k^i)) = \text{ord}_i\left(\prod_{\substack{q \in \overline{L}_G \\ q \not\leq m}} w_G(q)\right).$$

Since this holds for all i with $\mathcal{O}_i \neq \{0\}$, we get the formula. □

Next corollary gives a characterization of a squarefree ideal in terms of w_I .

Corollary 3.2. *A monomial ideal $I \subset S$ is squarefree if and only if $w_I(m)$ is squarefree for every $m \in \bar{L}_I$ and $\gcd(w_I(m), w_I(m')) = 1$ for all $m, m' \in \bar{L}_I, m \neq m'$.*

Proof. If $w_I(m)$ is squarefree for each $m \in \bar{L}_I$ and $\gcd(w_I(m), w_I(m')) = 1$ for $m \neq m'$, then I is squarefree, since by Proposition 3.1 every generator of I is a product of different monomials $w_I(m)$ for some $m \in \bar{L}_I$. On the other hand, if I is squarefree, then the lcm of all generators is a squarefree monomial. But this is the product of all the $w_I(m)$, so the claimed properties follow. \square

As a consequence for natural numbers, Proposition 3.1 implies the following observation.

Remark 3.3. Let $a \in \mathbb{N}$ be a natural number, $a > 1$. Denote by \mathcal{D}_a the set of its divisors and for $b \in \mathcal{D}_a$ set

$$w_a(b) := \frac{1}{b} \gcd\{c \in \mathcal{D}_a : c > b\} \in \mathbb{N}.$$

For $b \in \mathcal{D}_a$ it holds that

$$b = \prod_{\substack{d \in \mathcal{D}_a \\ b \nmid d}} w_a(d),$$

where we regard 1 as the product over the empty set.

We now come to our first main result. Using the inversion formula of Proposition 3.1, we can give a complete description of those pairs $(L, w : \bar{L} \rightarrow S)$ that come from a monomial ideal.

Theorem 3.4. *Let L be a finite join-semilattice. Let further $w : \bar{L} \rightarrow S$ be a map that assigns a monomial to each element of \bar{L} .*

- (1) *There exists a finite set $G \subset S$ of monomials with $L \cong L_G$ and $w = w_G$ if and only if the following two conditions are satisfied:*
 - (a) $\gcd(w(M), w(M')) = 1_{\mathbb{K}}$ for incomparable $M, M' \in L$.
 - (b) $w(M) \neq 1_{\mathbb{K}}$ if $M \in L$ is meet-irreducible and $w(\hat{1}_L) = 1_{\mathbb{K}}$.
- (2) *If the conditions of (1) are satisfied, then L_G (as set of monomials) is uniquely determined and its elements are given by the formula*

$$m_M = \prod_{\substack{Q \in \bar{L} \\ Q \not\leq M}} w(Q) \tag{3.1}$$

for each $M \in L$. In particular, the ideal generated by G is uniquely determined.

This theorem allows the very simple construction of ideals with a given lcm-semilattice. Moreover, considering the different possible maps $w : \bar{L} \rightarrow S$ one gets an overview over the class of all monomial ideals with a fixed lcm-semilattice.

Remark 3.5. The set G in part (1) is not uniquely determined itself, because there may be several sets of monomials generating the same L_G . For example consider the sets $G_1 = \{X, Y\}$ and $G_2 = \{X, Y, XY\}$. Then $L_{G_1} = L_{G_2}$.

Proof of Theorem 3.4. First, we prove that the conditions given in (1) are sufficient. Assume that (1a) and (1b) are true and consider $\psi : L \longrightarrow S$ defined by

$$\psi(M) := \prod_{\substack{Q \in \bar{L} \\ Q \not\leq M}} w(Q).$$

Set $G := \{\psi(M) : M \in L\} \subset S$. We are going to show that $L \cong L_G$ via ψ , but we need some preparations first.

By assumption (1a), for each variable X_i such that $X_i \mid \psi(\hat{1}_L)$, the set of $F \in \bar{L}$ such that $X_i \mid w(F)$ forms a chain $F_1^i < F_2^i < \dots < F_{r_i}^i$. Set $F_{r_i+1}^i := \hat{1}_L$ for $1 \leq i \leq n$, where $r_i = 0$ if $X_i \nmid \psi(\hat{1}_L)$. For $M \in \bar{L}$ let $s(i, M)$ be the minimal index k , such that $F_k^i \geq M$. Then

$$\text{ord}_i(\psi(M)) = \text{ord}_i\left(\prod_{j=1}^{s(i, M)-1} w(F_j^i)\right). \quad (3.2)$$

In particular, if $\psi(M) \mid \psi(M')$, then $s(i, M) \leq s(i, M')$ for $1 \leq i \leq n$, because of the inequality $\text{ord}_i(\psi(M)) \leq \text{ord}_i(\psi(M'))$.

Let $M, M' \in L$. We claim that $M \leq M'$ if and only if $\psi(M) \mid \psi(M')$. It is clear from the definition that $M \leq M'$ implies $\psi(M) \mid \psi(M')$, so assume that $\psi(M) \mid \psi(M')$. Every non-maximal element in a finite semilattice is the meet of the set of meet-irreducible elements greater than or equal to it (see Lemma 2.1). So, in order to show $M \leq M'$, we may prove the following: Each meet-irreducible element P which is greater than or equal to M' is also greater than or equal to M . So consider such an element P . By assumption (1b), $w(P) \neq 1$, so there exists an index i such that $X_i \mid w(P)$.

Then there exists a k such that $P = F_k^i$ (where $1 \leq k \leq r_i$). Now $M' \leq P$ implies that $s(i, M') \leq k$. But as remarked above, the fact that $\psi(M) \mid \psi(M')$ implies that $s(i, M) \leq s(i, M') \leq k$, hence $P \geq M$.

It follows that ψ is injective, as $\psi(M) = \psi(M')$ implies $\psi(M) \mid \psi(M') \mid \psi(M)$ and thus $M \leq M' \leq M$.

Further, we claim that $\psi(M \vee M')$ equals the lcm of $\psi(M)$ and $\psi(M')$. For this, first note that $P \geq M \vee M'$ if and only if $P \geq M$ and $P \geq M'$. This implies that $s(i, M \vee M') = \max(s(i, M), s(i, M'))$ for all i . Therefore

$$\text{ord}_i(\psi(M \vee M')) = \max(\text{ord}_i(\psi(M)), \text{ord}_i(\psi(M'))) = \text{ord}_i(\text{lcm}(\psi(M), \psi(M')))$$

for all i , hence $\psi(M \vee M') = \text{lcm}(\psi(M), \psi(M'))$.

Summarizing, we have shown that ψ is an injective map $L \longrightarrow G$ which preserves the join. The latter implies that G is closed under taking the lcm, i.e. $G = L_G$. Hence ψ is also surjective onto L_G and thus induces an isomorphism $L \longrightarrow L_G$.

It remains to show that $w(M) = w_G(\psi(M))$ for all $M \in \bar{L}$. By definition of w_G , we have to show that $\gcd(\psi(P) : P > M) = \psi(M)w(M)$ if $M \neq \hat{0}$ and $\gcd(\psi(P) : P > M) = w(M)$ for $M = \hat{0}$. We handle both cases together by proving that

$$\text{ord}_i(\gcd(\psi(P) : P > M)) = \text{ord}_i(\psi(M)) + \text{ord}_i(w(M))$$

for each i , where we adopt the convention that $\psi(\hat{0}) = 1_{\mathbb{K}}$. We compute

$$\begin{aligned} \text{ord}_i(\gcd(\psi(P) : P > M)) &= \min\{\text{ord}_i(\psi(P)) : P > M\} \\ &= \min\{\text{ord}_i\left(\prod_{j=1}^{s(i,P)-1} w(F_j^i)\right) : P > M\} \\ &= \text{ord}_i\left(\prod_{j=1}^{k-1} w(F_j^i)\right) \end{aligned}$$

where $k := \min\{s(i, P) : P > M\}$. We compute further:

$$\begin{aligned} k &= \min\{s(i, P) : P > M\} = \min\{\min\{j : F_j^i \geq P\} : P > M\} \\ &= \min\{j : F_j^i > M\} \\ &= \begin{cases} s(i, M) + 1 & \text{if } M = F_{s(i, M)}^i, \\ s(i, M) & \text{otherwise.} \end{cases} \end{aligned}$$

Remark that in the second case it holds that $\text{ord}_i(w(M)) = 0$ (otherwise $M = F_{s(i, M)}^i$ by assumption (1a)).

Recall that $\text{ord}_i(\psi(M)) = \text{ord}_i(\prod_{j=1}^{s(i, M)-1} w(F_j^i))$. So we conclude that

$$\begin{aligned} \text{ord}_i(\gcd(\psi(P) : P > M)) &= \\ &= \begin{cases} \text{ord}_i(\prod_{j=1}^{s(i, M)-1} w(F_j^i)) + \text{ord}_i(w(F_{s(i, M)}^i)) & \text{if } M = F_{s(i, M)}^i, \\ \text{ord}_i(\prod_{j=1}^{s(i, M)-1} w(F_j^i)) & \text{otherwise} \end{cases} \\ &= \text{ord}_i(\psi(M)) + \text{ord}_i(w(M)). \end{aligned}$$

Next, we prove that the conditions given in (1) are necessary. Let $G \subset S$ be a finite set of monomials such that there exists an isomorphism $\varphi : L \rightarrow L_G$ such that $w = w_G \circ \varphi$.

(1a) Let M and M' be two elements of L such that $\gcd(w(M), w(M')) \neq 1$. Then $m = \varphi(M)$ and $m' = \varphi(M')$ are two elements of L_G such that $\gcd(w_G(m), w_G(m')) \neq 1$. There exists a variable X_i such that $X_i \mid \gcd(w_G(m), w_G(m'))$. Then $\text{ord}_i(p) > \text{ord}_i(m)$ for all $p \in L_G$ with $p > m$ and $\text{ord}_i(p) > \text{ord}_i(m')$ for all $p \in L_G$ with $p > m'$. But $m \vee m' \geq m$, $m \vee m' \geq m'$ and $\text{ord}_i(m \vee m') = \max(\text{ord}_i(m), \text{ord}_i(m'))$. So, if we assume $m \vee m' > m$ and $m \vee m' > m'$, then $X_i \nmid w_G(m)$ or $X_i \nmid w_G(m')$, contradicting our choice of X_i . This together implies that $m \vee m'$ equals either m or m' , hence $m \leq m'$ or $m \geq m'$, which in turn implies $M \leq M'$ or $M \geq M'$.

(1b) First note that $w(\hat{1}_L) = w_G(\hat{1}) = 1_{\mathbb{K}}$. Let M be a meet-irreducible element of L . Then $m = \varphi(M)$ is a meet-irreducible element of L_G and thus covered by exactly one element p , hence $w(M) = w_G(m) = \frac{p}{m} \neq 1$.

For the uniqueness statement of (2), it is sufficient to show that

$$m = \prod_{\substack{Q \in \bar{L} \\ Q \not\leq \varphi^{-1}(m)}} w(Q)$$

for every $m \in L_G$, because this implies that the elements of L_G are determined by the values of w . By Proposition 3.1 we have

$$m = \prod_{\substack{q \in \bar{L}_G \\ q \not\leq m}} w_G(q) = \prod_{\substack{Q \in \bar{L} \\ Q \not\leq \phi^{-1}(m)}} w_G(\phi(Q)) = \prod_{\substack{Q \in \bar{L} \\ Q \not\leq \phi^{-1}(m)}} w(Q).$$

□

The next corollary is known for the case of atomistic (semi-)lattices (see Phan [Pha05]) and may also be known for finite (semi-)lattices in general. Since we could not find a reference to it, we present it below.

Corollary 3.6. *Every finite join-semilattice can be realized as lcm-semilattice of a set of monomials.*

Proof. Let $S = \mathbb{K}[X_M : M \in L \text{ meet-irreducible}]$ be the polynomial ring with variables indexed by the set of meet-irreducible elements of L . Then the map defined by

$$w(M) := \begin{cases} X_M & \text{if } M \text{ is meet-irreducible} \\ 1 & \text{otherwise} \end{cases}$$

clearly satisfies the conditions of part (1) of Theorem 3.4. □

The following result will not be used in the sequel. However, we consider it as being of independent interest.

Corollary 3.7. *Let $I \subset \mathbb{K}[X_1, \dots, X_n]$ and $I' \subset \mathbb{K}[X_1, \dots, X_{n'}]$ be two squarefree monomial ideals in two (possibly different) polynomial rings. Assume that there exists an isomorphism of lcm-semilattices $\delta : L_I \rightarrow L_{I'}$, such that $\deg w_I(m) = \deg w_{I'}(\delta(m))$ for all $m \in \bar{L}_I$. Then $n = n'$ and $I = I'$ up to a permutation of the variables.*

Proof. By Corollary 3.2 and our assumption that

$$\deg w_I(m) = \deg w_{I'}(\delta(m)),$$

it follows that $n = n'$ and $w_I(m) = w_{I'}(\delta(m))$ up to a relabeling of the variables. Now the claim follows from the uniqueness part of Theorem 3.4. □

4. INVARIANTS AND SURJECTIVE JOIN-PRESERVING MAPS

4.1. The structure of surjective join-preserving maps. In this Subsection we prove some structural results on surjective join-preserving maps; these will be needed in the sequel. The first two structural lemmata will be useful in Subsection 4.2.

Let $\phi : L \rightarrow L'$ be a surjective join-preserving map of finite join-semilattices. We define $\phi^\dagger : L' \rightarrow L$ as $\phi^\dagger(a) := \bigvee \phi^{-1}(a)$.

Lemma 4.1. *The map ϕ^\dagger has the following properties:*

- (1) $\phi \circ \phi^\dagger : L' \rightarrow L'$ is the identity and ϕ^\dagger is (thus) injective.
- (2) ϕ^\dagger is monotonic, i.e. $a \leq b$ implies $\phi^\dagger(a) \leq \phi^\dagger(b)$ for $a, b \in L'$.
- (3) For $\alpha \in L$ and $b \in L'$, it holds that $\phi(\alpha) \leq b$ if and only if $\alpha \leq \phi^\dagger(b)$.

Proof. The first claim is immediate from the fact that ϕ preserves joins. For the second, note that $\phi(\phi^\dagger(a) \vee \phi^\dagger(b)) = \phi(\phi^\dagger(a)) \vee \phi(\phi^\dagger(b)) = a \vee b$ for any $a, b \in L'$. Thus $\phi^\dagger(a) \vee \phi^\dagger(b)$ is contained in the preimage of $a \vee b$ and hence $\phi^\dagger(a) \vee \phi^\dagger(b) \leq \phi^\dagger(a \vee b)$. Now assume that $a \leq b$. Then

$$\phi^\dagger(a) \leq \phi^\dagger(a) \vee \phi^\dagger(b) \leq \phi^\dagger(a \vee b) = \phi^\dagger(b).$$

For the last claim, note that $\alpha \leq \phi^\dagger(b)$ implies $\phi(\alpha) \leq \phi(\phi^\dagger(b)) = b$, and $\phi(\alpha) \leq b$ implies $\alpha \leq \phi^\dagger(\phi(\alpha)) \leq \phi^\dagger(b)$ (since ϕ^\dagger is monotonic). \square

Lemma 4.2. *Let $G \subset S$ and $G' \subset S'$ be two finite sets of squarefree monomials in two polynomial rings. Assume that there exists a surjective join-preserving map $\phi : L_G \rightarrow L_{G'}$ such that $\deg w_G(\phi^\dagger(m')) \geq \deg w_{G'}(m')$ for all $m' \in \bar{L}_{G'}$. Then there exists a ring homomorphism $\Psi : S \rightarrow S'$ sending a subset of the variables injectively to the variables of S' and the other variables to 1. This map satisfies $\Psi(m) = \phi(m)$ for $m \in L_G$.*

Proof. As G and G' consist only of squarefree monomials, it holds that all values of w_G and $w_{G'}$ are squarefree and pairwise coprime by Corollary 3.2.

We define Ψ as follows: For every $m' \in \bar{L}_{G'}$, choose $\deg w_{G'}(m')$ many variables dividing $w_G(\phi^\dagger(m'))$ and let them map bijectively to the variables dividing $w_{G'}(m')$. The remaining variables of S are mapped to one. By construction, for $m \in \bar{L}_G$ it holds that

$$\Psi(w_G(m)) = \begin{cases} w_{G'}(m') & \text{if } m \in \phi^\dagger(\bar{L}_{G'}) \text{ and } m = \phi^\dagger(m'); \\ 1 & \text{if } m \notin \phi^\dagger(\bar{L}_{G'}). \end{cases}$$

Using Proposition 3.1 we conclude that

$$\Psi(m) = \Psi\left(\prod_{\substack{q \in \bar{L}_G \\ q \nmid m}} w_G(q)\right) = \prod_{\substack{q \in \bar{L}_G \\ q \nmid m}} \Psi(w_G(q)) = \prod_{\substack{q' \in \bar{L}_{G'} \\ q' \nmid \phi(m)}} w_{G'}(q') = \phi(m).$$

where $m \in L_G$. For the third equality, we used part (3) of Lemma 4.1. \square

The next two structural lemmata will be used in Subsection 5.1. Fix a meet-irreducible element $a \in L$ and let $a_+ \in L$ denote the unique element covering it. We consider the equivalence relation \sim_a on L defined by setting $a \sim_a a_+$ and any other element is equivalent only to itself.

Lemma 4.3. *There is a natural join-semilattice structure on L/\sim_a , such that the canonical surjection $\pi_a : L \rightarrow L/\sim_a$ preserves the join. Moreover, if L is atomistic and a is not an atom, then L/\sim_a is atomistic.*

Proof. Let \bar{b} denote the equivalence class of an element $b \in L$. We define $\bar{b} \vee \bar{c} := \overline{b \vee c}$. To show that this is well-defined we have to prove that $b_1 \sim_a b_2$ and $c_1 \sim_a c_2$ implies $b_1 \vee c_1 \sim_a b_2 \vee c_2$. But this follows easily from the fact $a \vee b \sim_a a_+ \vee b$ for all $b \in L$.

The \vee -operation on L/\sim_a inherits associativity, commutativity and idempotency from the join of L , so it endows L/\sim_a with the structure of a join-semilattice, cf. [DP02, Thm. 2.10]¹. It is clear that π_a preserves this join. The last statement is also clear as π_a is a bijection on the atoms. \square

¹The result in [DP02] is stated for lattices only, but this is easily seen to hold in our situation.

Lemma 4.4. *Let L, L' be finite join-semilattices and $\phi : L \rightarrow L'$ a join-preserving map.*

- (1) *If ϕ is not injective, then there exists a meet-irreducible element $a \in L$ such that $\phi(a) = \phi(a_+)$.*
- (2) *If $\phi(a) = \phi(a_+)$ for some meet-irreducible element $a \in L$, then ϕ factors through L / \sim_a .*

Proof. (1) There exists a maximal element $b \in L$ such that the pre-image of $\phi(b)$ has at least two elements, that is $|\phi^{-1}(\phi(b))| > 1$. Choose another element $b' \in \phi^{-1}(\phi(b))$, $b' \neq b$. Then $b' < b$ by maximality, as $\phi(b \vee b') = \phi(b) \vee \phi(b') = \phi(b)$. It is easy to see that the interval $[b', b]$ is mapped to $\phi(b)$, so we may choose $a \in L$ such that $\phi(a) = \phi(b)$ and a is covered by b . We claim that this a is meet-irreducible. Assume to the contrary that there exists another element $c \neq b$ covering a . Then

$$\phi(c) = \phi(c \vee a) = \phi(c) \vee \phi(a) = \phi(c) \vee \phi(b) = \phi(c \vee b)$$

As $b < b \vee c$, it follows from our choice of b that $c = c \vee b$ and thus $c > b$, a contradiction.

(2) It is clear that ϕ factors through L / \sim_a set-theoretically, i.e. there exists a map $\bar{\phi} : L / \sim_a \rightarrow L'$ such that $\phi = \bar{\phi} \circ \pi_a$. So we only need to show that $\bar{\phi}$ preserves the join. This is an easy computation:

$$\bar{\phi}(\overline{b \vee c}) = \bar{\phi}(\overline{b \vee c}) = \phi(b \vee c) = \phi(b) \vee \phi(c) = \bar{\phi}(\bar{b}) \vee \bar{\phi}(\bar{c})$$

for $b, c \in L$. □

4.2. Stanley projective dimension and surjective join-preserving maps. In this Subsection we prove the following theorem, which is the main result of this paper.

Theorem 4.5. *Let $J \subsetneq I \subset S$ and $J' \subsetneq I' \subset S'$ be four monomial ideals in two (possibly) different polynomial rings. Assume that the ideals have generating sets $G_J \subset J, G_I \subset I, G_{J'} \subset J', G_{I'} \subset I'$ such that there exists a surjective join-preserving map $\delta : L_{G_I \cup G_J} \rightarrow L_{G_{I'} \cup G_{J'}}$ of the lcm-semilattices, which maps L_{G_J} surjectively onto $L_{G_{J'}}$. Then*

$$\text{spdim}_S I/J \geq \text{spdim}_{S'} I'/J'.$$

Moreover, if δ is bijective, then

$$\text{spdim}_S I/J = \text{spdim}_{S'} I'/J'.$$

In the special case that $I = S$ and $I' = S'$, i.e. $I/J = S/J$, it is enough to have a surjective join-preserving map $\phi : L_{G(J)} \rightarrow L_{G(J')}$. We then can choose $1_{\mathbb{K}}$ as generator for both S and S' and canonically extend ϕ to a map $L_{G_I \cup G_J} \rightarrow L_{G_{I'} \cup G_{J'}}$ satisfying the assumption of the theorem. Similarly, in the case $J = J' = (0)$, it is enough to have a surjective join-preserving map $\phi : L_{G(I)} \rightarrow L_{G(I')}$.

Before we give the proof of Theorem 4.5 we prepare two lemmata. The first one is a slight generalization of Cimpoeaş [Cim08, Lemma 2.2]. It could be derived from [IMF14, Proposition 5.2], but for the reader's convenience we provide the (easy) proof.

Lemma 4.6. *Let $J \subset I \subset S[Y] = \mathbb{K}[X_1, \dots, X_n, Y]$ be two squarefree monomial ideals. Let $J' \subset I' \subset S$ be the images of J and I under the map sending Y to 1. Then we have*

$$\text{spdim}_{S[Y]} I/J \geq \text{spdim}_S I'/J'.$$

Proof. Let $M := I/J$ and let $M_{>0} \subset M$ be the $S[Y]$ -submodule of those elements having positive Y -degree. Every Stanley decomposition of M restricts to a Stanley decomposition of $M_{>0}$, hence $\text{spdim}_{S[Y]} M \geq \text{spdim}_{S[Y]} M_{>0}$.

On the other hand, we have

$$M_{>0} = I/J \cap ((Y)/J) = (I \cap (Y))/(J \cap (Y)) \cong (I : Y)/(J : Y) = (I : Y^\infty)/(J : Y^\infty),$$

where for the last equality we use that I and J are squarefree. But $I : Y^\infty = I' \otimes_S S[Y]$ and the same holds for J , hence $M_{>0} \cong I'/J' \otimes_S S[Y]$. By [IMF14, Proposition 5.1] we conclude that $\text{spdim}_{S[Y]} M_{>0} = \text{spdim}_S I'/J'$ and the claim follows. \square

The second lemma comprises the main part of the proof of the theorem.

Lemma 4.7. *Let $G_1, G_2 \subset S = \mathbb{K}[X_1, \dots, X_n]$ be two sets of squarefree monomials. Let $I := (G_1)$, $J := (G_2)$ the ideals generated by G_1 and G_2 and assume that $J \subsetneq I$. Let $m \in \bar{L}_{G_1 \cup G_2}$ be a fixed element. Then there exist two other sets of squarefree monomials $G'_1, G'_2 \subset S[Y]$ in one additional variable Y , such that the following holds:*

- (1) *There is an isomorphism $\delta : L_{G_1 \cup G_2} \longrightarrow L_{G'_1 \cup G'_2}$ of the lcm-semilattices, such that for every $c \in \bar{L}_{G_1 \cup G_2}$ it holds that*

$$\deg w_{G'_1 \cup G'_2}(\delta(c)) := \begin{cases} \deg w_{G_1 \cup G_2}(c) & \text{if } c \neq m, \\ \deg w_{G_1 \cup G_2}(c) + 1 & \text{if } c = m. \end{cases}$$

Moreover, δ maps L_{G_2} to $L_{G'_2}$.

- (2) $J' \subsetneq I' \subset S[Y]$, where $I' := (G'_1)$ and $J' := (G'_2)$.
 (3) $\text{spdim}_{S[Y]} I'/J' = \text{spdim}_S I/J$.

Proof. Consider the map of monomials $\delta : S \longrightarrow S[Y]$ given by

$$\delta(c) = \begin{cases} c & \text{if } c \mid m, \\ Yc & \text{if } c \nmid m. \end{cases}$$

We define G'_1 and G'_2 as the images of G_1 resp. G_2 under this map. It is easy to see that δ preserves the lcm of monomials and thus restricts to a join-preserving map $\delta : L_{G_1 \cup G_2} \longrightarrow L_{G'_1 \cup G'_2}$. δ is clearly injective and, as every element of $L_{G'_1 \cup G'_2}$ is an lcm of elements in $G'_1 \cup G'_2$, also surjective on $L_{G'_1 \cup G'_2}$. Moreover, it follows from the definitions that

$$w_{G'_1 \cup G'_2}(m) = \frac{1}{m} \gcd\{p \in L_{G'_1 \cup G'_2} : p > m\} = Y w_{G_1 \cup G_2}(m)$$

and $w_{G'_1 \cup G'_2}(\delta(c)) = w_{G_1 \cup G_2}(c)$ for every other $c \in \bar{L}_{G_1 \cup G_2}$. Part (1) of the lemma is then proven. Part (2) follows straight from the fact that δ is monotonic.

(3) The inequality “ \geq ” follows from Lemma 4.6, as J and I are the images of J' and I' under sending Y to 1. So we only need to prove the other inequality, which is equivalent to $\text{sdepth}_{S[Y]} I'/J' \geq \text{sdepth}_S I/J + 1$. To simplify the notation, we set $0_k := (0, \dots, 0) \in \mathbb{N}^k$ and $1_k := (1, \dots, 1) \in \mathbb{N}^k$ for $k \in \mathbb{N}$. After relabeling of the variables, we may assume that

$m = X_1 X_2 \cdots X_l$ for some $1 \leq l \leq n$. Consider the map $\phi : \mathbb{N}^l \times \mathbb{N}^{n-l} \times \mathbb{N} \longrightarrow \mathbb{N}^l \times \mathbb{N}^{n-l}$ defined by

$$\phi(h_1, h_2, x) := \begin{cases} (h_1, 0_{n-l}) & \text{if } x = 0, \\ (h_1, h_2) & \text{if } x > 0. \end{cases}$$

It is easy to see that ϕ is order preserving. Let $\Phi : S[Y] \longrightarrow S$ be the corresponding map on monomials. We claim that $I' = \Phi^{-1}(I)$, $J' = \Phi^{-1}(J)$ and that Φ changes the Stanley depth by 1 with respect to $1_{n+1} \in \mathbb{N}^{n+1}$ and $1_n \in \mathbb{N}^n$ (see Definition 2.7).

For the first claim, it suffices to consider I and I' . Note that $\Phi(G'_1) = G_1$ and thus $I' \subset \Phi^{-1}(I)$. For the other inclusion, consider a minimal element $e = (h_1, h_2, x) \in \Phi^{-1}(I)$. By minimality, it follows that $x \in \{0, 1\}$. There are two cases:

- (i) If $x = 1$, then $\phi(e) = (h_1, h_2)$. This is a minimal element of I , because if $I \ni (h'_1, h'_2) < (h_1, h_2)$, then $(h'_1, h'_2, 1) \in \phi^{-1}(h'_1, h'_2) \subset \Phi^{-1}(I)$, contradicting the minimality of e . But now $(h_1, h_2, 1)$ is by definition a generator of I' .
- (ii) If $x = 0$, then $h_2 = 0_{n-l}$. Again, $(h_1, 0_{n-l})$ is a minimal element of I , because otherwise $(h'_1, 0_{n-l}, 0)$ would be a smaller element in $\Phi^{-1}(I)$ as above. So again, $(h_1, 0_{n-l}, 0)$ is a generator of I' .

So we conclude that $I' \supseteq \Phi^{-1}(I)$ and thus $I' = \Phi^{-1}(I)$.

For the second claim, it is enough to consider the map $\psi : \mathbb{N}^k \times \mathbb{N} \longrightarrow \mathbb{N}^k$, defined by

$$\psi(h, x) := \begin{cases} 0_k & \text{if } x = 0; \\ h & \text{if } x > 0. \end{cases}$$

where $k := n - l$. It is clear that $\phi = (id_{\mathbb{N}^l}, \psi)$. One easily checks that

$$\psi^{-1}(h) = \begin{cases} \{(h, x) : x > 0\} & \text{if } h \neq 0_k; \\ \{(w, 0) : w \in \mathbb{N}^k\} & \text{if } h = 0_k. \end{cases}$$

Consider an interval $[a, b] \subset [0_k, 1_k]$. It follows that

$$\psi^{-1}([a, b]) \cap [0_{k+1}, 1_{k+1}] = \begin{cases} [(a, 1), (b, 1)] & \text{if } a \neq 0_k; \\ [(0_k, 0), (1_k, 0)] \cup [(0_k, 1), (b, 1)] & \text{if } a = 0_k, b \neq 0_k, 1_k; \\ [(0_k, 0), (1_k, 1)] & \text{if } a = 0_k, b = 1_k; \\ [(0_k, 0), (1_k, 0)] & \text{if } a = 0_k, b = 0_k. \end{cases}$$

In each case, the Stanley depth is increased (at least) by one. The only case that needs a closer look is the second. Here, $\rho((1_k, 0)) = k$, but $\rho(b) < k$, because $b \neq 1_k$, so this also increases the Stanley depth. Therefore ψ increases the Stanley depth by 1, and so does ϕ by Lemma 2.9. We conclude that the claim follows from Proposition 2.8. \square

Proof of Theorem 4.5. We may assume that all involved ideals are squarefree by Theorem 2.6. Next, after repeated application of Lemma 4.7 to G_J and G_I we may also assume that

$$\deg w_{G_I \cup G_J}(\phi^\dagger(m)) \geq \deg w_{G'_I \cup G'_J}(m)$$

for all $m \in \bar{L}_{G'_I \cup G'_J}$. It follows from Lemma 4.2 that I' and J' are the images of I and J under a homomorphism sending some of the variables to 1. Here we use that δ respects

the subsemilattice generated by the generators of J . Now the claim follows from Lemma 4.6. \square

The following example illustrates the effectiveness of Theorem 4.5 for a situation where, for example, the use of Herzog et al. [HVZ09, Theorem 2.1] would lead to extensive computations.

Example 4.8. The ideals

$$I_1 := (yzv, xzv, x^2yv, x^3yz)$$

and

$$I_2 := (yzv, xzv, xy^2v, x^2y^2z)$$

have isomorphic lcm-semilattices and thus have the same Stanley depth. However, it is clear that the characteristic posets of [HVZ09, Theorem 2.1] are not related. In Figure 1 the lcm-semilattices of the two ideals are depicted. In the rightmost diagram the values of w_I are shown. The left letters correspond to I_1 and the right letters in uppercase font correspond to I_2 .

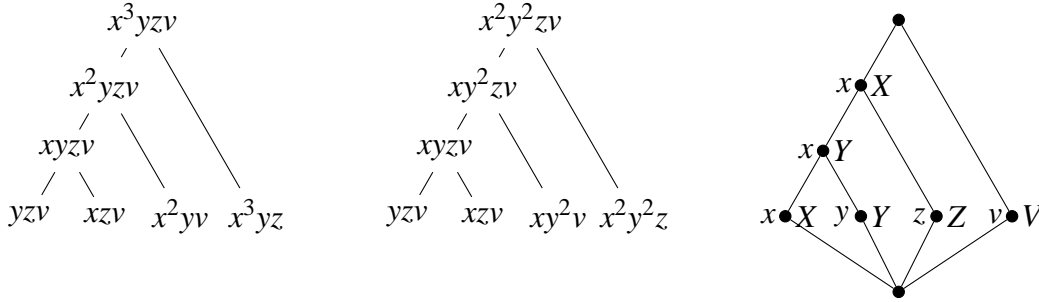


FIGURE 1. The lcm-semilattices of two monomial ideals.

4.3. Projective dimension and surjective join-preserving maps. In this subsection, we provide the analogue of Theorem 4.5 for the usual projective dimension. Let $I \subsetneq S$ and $I' \subsetneq S'$ be two monomial ideals, such that there exists a surjective join-preserving map $\phi : L_I \rightarrow L_{I'}$. If we assume further that ϕ is bijective on the generators, then Theorem 3.3 of [GPW99] immediately implies that

$$\text{pdim}_S S/I \geq \text{pdim}_{S'} S'/I'. \quad (4.1)$$

However, we would like to have a result in the same generality as Theorem 4.5. Thus, we prove the following extension of the inequality (4.1).

Theorem 4.9. *Let $J \subsetneq I \subset S$ and $J' \subsetneq I' \subset S'$ be four monomial ideals in two (possibly) different polynomial rings. Assume that the ideals have generating sets $G_J \subset J, G_I \subset I, G_{J'} \subset J', G_{I'} \subset I'$ such that there exists a surjective join-preserving map $\delta : L_{G_I \cup G_J} \rightarrow L_{G_{I'} \cup G_{J'}}$ of lcm-semilattices, which maps L_{G_J} surjectively onto $L_{G_{J'}}$. Then*

$$\text{pdim}_S I/J \geq \text{pdim}_{S'} I'/J'.$$

Moreover, if δ is bijective, then

$$\text{pdim}_S I/J = \text{pdim}_{S'} I'/J'.$$

Proof. The idea of the proof is the same as in the proof of [GPW99, Theorem 3.3]. We split the proof into several steps. To simplify notation, set $G_1 := G_I \cup G_J$ and $G_2 := G_J$.

Step 1. Define $\delta(G_1) := \{\delta(g) : g \in G_1\}$ as a *multiset*, i.e. if several elements of G_1 have the same image under δ , then this image is contained more than once in $\delta(G_1)$. Note that the ideal generated by $\delta(G_1)$ is I' and $L_{G'_I \cup G'_J} = L_{\delta(G_1)}$, where the latter is the lcm-semilattice of the underlying set of G_1 ; and the same holds for G_2 .

By replacing $G_{I'}$ and $G_{J'}$ with $\delta(G_1)$ and $\delta(G_2)$, we now can make the additional assumption that δ is bijective on the given (multi-)sets of generators of I' and J' .

Step 2. Let T_{G_1} denote the Taylor complex defined by G_1 . For its definition see for example Eisenbud [Eis95, Ex. 17.1] or Miller and Sturmfels [MS05, 4.3.2]. Recall that T_{G_1} is a free resolution of S/I and that the i -th module of T_{G_1} has a basis indexed by the i -subsets of G_1 . Moreover, $G_2 \subset G_1$ implies that T_{G_2} is a subcomplex of T_{G_1} . Let $T_{G_1/G_2} := T_{G_1}/T_{G_2}$ denote the quotient complex. It follows from the construction of the Taylor complex that every module of T_{G_2} is a direct summand of the corresponding module of T_{G_1} , hence T_{G_1/G_2} is again a complex of free S -modules. As T_{G_1} and T_{G_2} are free resolutions of S/I and S/J resp., it can be seen from the long exact sequence in homology (cf. Rotman [Rot09, Theorem 6.10]) that T_{G_1/G_2} is a free resolution of I/J . See also Lemma 3.3.2 in Orlik and Welker [OW07].

Finally, note that everything we wrote in this paragraph still holds if we allow G_1 and G_2 to be multisets.

Step 3. It follows from the definition of T_G that $\delta(T_{G_1}) = T_{\delta(G_1)}$ where $\delta(T_{G_1})$ is the *relabeling* of T_{G_1} induced by δ as introduced in [GPW99, Construction 3.2].

Moreover, as δ maps L_{G_2} onto $L_{\delta(G_2)}$, it further holds that $\delta(T_{G_2}) = T_{\delta(G_2)}$ and (thus) $\delta(T_{G_1/G_2}) = T_{\delta(G_1)/\delta(G_2)}$ is a free resolution of I'/J' .

From now on, the argument is completely analogous to the proof of [GPW99, Theorem 3.3]. By [Eis95, Theorem 20.2], $T_{G_1/G_2} = P \oplus F$, where F is a minimal free resolution of I/J and P is a direct sum of complexes of the form $0 \rightarrow S \rightarrow S \rightarrow 0$. As relabeling is compatible with direct sums, it follows that $\delta(T_{G_1/G_2}) = \delta(P) \oplus \delta(F)$ and $\delta(P)$ is still acyclic. Hence $\delta(F)$ is an (in general not minimal) free resolution of I'/J' with length $\text{pdim}_S I/J$. So the claim is proven. \square

Corollary 4.10. *With the notation from above, if the map δ is bijective, then*

- (1) $\text{spdim}_S I/J = \text{spdim}_{S'} I'/J'$;
- (2) $\text{pdim}_S I/J = \text{pdim}_{S'} I'/J'$;
- (3) $\text{sdepth}_S I/J - \text{depth}_S I/J = \text{sdepth}_{S'} I'/J' - \text{depth}_{S'} I'/J'$.

Remark 4.11. The assumptions of the corollary are satisfied in the case that I' resp. J' are the polarization of I resp. J . Therefore the last part of the corollary is a generalization of the authors' result on polarization [IKMF14, Theorem 4.4]. This does not diminish the importance of [IKMF14, Theorem 4.4], since it is required in the proof of Theorem 4.5.

In view of part (3) of Corollary 4.12, one may ask how the quantity $\text{sdepth}_S I/J - \text{depth}_S I/J$ behaves under surjective maps of the lcm-semilattice. In general there is no inequality, as can be seen in the following example.

Example 4.12. Consider the maximal ideal $I_1 = (X_1, \dots, X_k) \subset S = \mathbb{K}[X_1, \dots, X_k]$. It is well-known that $\text{sdepth}_S S/I_1 = \text{depth}_S S/I_1 = 0$ and thus $\text{sdepth}_S S/I_1 - \text{depth}_S S/I_1 = 0$.

Moreover, consider the ideal $I_2 := (\frac{x_1 \cdots x_k}{x_i} : 1 \leq i \leq k) \subset S$. Its lcm-semilattice consists only of an antichain of k atoms, together with a maximal element. Thus every lcm-semilattice of an ideal with k generators can be mapped onto it. This holds in particular for the lcm-semilattice of the ideal $(x^{k-1}, x^{k-2}y, \dots, xy^{k-2}, y^{k-1}) \subset \mathbb{K}[x, y]$. So it follows from Theorem 4.5 and Theorem 4.9 that $\text{spdim}_S S/I_2 \leq 2$ and $\text{pdim}_S S/I_2 \leq 2$. On the other hand, it holds that $\text{pdim}_S S/I_2 \geq 2$ because I_2 is not principal and $\text{spdim}_S S/I_2 \geq 2$ by Proposition 5.2 below. So we can conclude that $\text{pdim}_S S/I_2 = \text{spdim}_S S/I_2 = 2$, hence $\text{sdepth}_S S/I_2 - \text{depth}_S S/I_2 = 0$.

If now $I' \subset S'$ is an arbitrary monomial ideal with k minimal generators, then there are surjective maps $L_{I_1} \rightarrow L_{I'}$ and $L_{I'} \rightarrow L_{I_2}$. Thus if $\text{sdepth}_{S'} S'/I' - \text{depth}_{S'} S'/I' \neq 0$, then this quantity is in general not monotonic under surjective maps. For example, one may take I' as any monomial ideal whose depth depends on the characteristic of the field.

5. APPLICATIONS

Theorems 4.5 and 4.9 are the sources for several applications to which this section is devoted. Essentially all inequalities for the Stanley projective dimension we derive in this section rely on Theorem 4.5. Using Theorem 4.9 one obtains with the same proof corresponding inequalities for the usual projective dimension. More generally, this holds for any invariant of an ideal or its lcm-semilattice which satisfies the conclusion of Theorem 4.5. These are, for example, cardinality, length, width, breadth, order dimension and interval dimension of the lcm-semilattice.

5.1. Bounds for the Stanley depth in terms of generators. For $k \in \mathbb{N}$ let $\mathfrak{B}(k)$ denote the semilattice of nonempty subsets of a k -element set. Note that $\mathfrak{B}(k)$ is the lcm-semilattice of an ideal generated by k variables and that $\overline{\mathfrak{B}}(k)$ is the boolean lattice on k atoms. First of all, we make the following simple observation.

Remark 5.1. For every atomistic semilattice L on k atoms, there exists a surjective join-preserving map $\phi : \mathfrak{B}(k) \rightarrow L$. The map ϕ may be constructed as follows. Let ϕ map the atoms of $\mathfrak{B}(k)$ bijectively on the atoms of L . For every other element $a \in \mathfrak{B}(k)$ we set $\phi(a) := \phi(a_1) \vee \cdots \vee \phi(a_l)$ where $a = a_1 \vee a_2 \vee \cdots \vee a_l$ is the unique (up to order) way to write a as a join of atoms.

We give a uniform proof of important results previously obtained by several authors providing bounds on the Stanley depth.

Proposition 5.2. *Let $k > 1$ and let $I \subset S$ be a monomial ideal with k minimal generators. Let further $\mathfrak{m}_k := (Y_1, \dots, Y_k) \subset S_k := \mathbb{K}[Y_1, \dots, Y_k]$ be the monomial maximal ideal on k generators. Then the following inequalities hold:*

- (1) $1 \leq \text{spdim}_S I \leq \text{spdim}_{S_k} \mathfrak{m}_k = \lfloor \frac{k}{2} \rfloor$
- (2) $2 \leq \text{spdim}_S S/I \leq \text{spdim}_{S_k} S_k/\mathfrak{m}_k = k$

Moreover, if I is a complete intersection, then the upper bounds are attained.

The assumption that $k > 1$ was introduced in order to avoid the zero module S/S . The upper bound for $\text{spdim}_S S/I$ was originally proven by Cimpoeaş [Cim09, Prop. 1.2] and

the upper bound for $\text{spdim}_S I$ was originally proven by Okazaki [Oka11], resp. in the squarefree case by Keller and Young [KY09]. For a complete intersection I the Stanley depth of I was originally determined by Shen [She09] and of S/I by Rauf [Rau07].

Proof of Proposition 5.2. The Stanley depth of the maximal ideal \mathfrak{m}_k was computed by Biró et al. in [BHK⁺10]. Moreover, the Stanley depth of $\mathbb{K} = S_k/\mathfrak{m}_k$ is zero. Thus the values of the upper bounds are known. The inequalities $\text{spdim}_S I \leq \text{spdim}_{S_k} \mathfrak{m}_k$ and $\text{spdim}_S S/I \leq \text{spdim}_{S_k} S/\mathfrak{m}_k$ follow from Theorem 4.5, since by Remark 5.1 there exists a surjective join-preserving map $L_{\mathfrak{m}_k} = \mathfrak{B}(k) \longrightarrow L_I$.

If I is a complete intersection, then $L_I \cong L_{\mathfrak{m}_k}$, therefore $\text{spdim}_S I = \text{spdim}_{S_k} \mathfrak{m}_k$ and $\text{spdim}_S S/I = \text{spdim}_{S_k} S/\mathfrak{m}_k$ by Theorem 4.5.

For the lower bound, note that every ideal in n variables with more than one minimal generator has Stanley depth less than n . Moreover, if I has at least 2 generators, then there exists a monomial m in $S \setminus I$ and two coprime monomials $n_1, n_2 \in S$ such that $mn_1, mn_2 \in I$. This implies that S/I has an associated prime of height at least 2 and thus the Stanley depth of S/I is at most $n - 2$. \square

In the next result we characterize the case of equality for the upper bound in part (2) of Proposition 5.2. Recall that monomial complete intersections can be characterized as those ideals I whose number of generators equals the projective dimension of S/I . In this sense, the last sentence of the following theorem extends the result on the Stanley depth of complete intersections [She09].

Theorem 5.3. *Let $I \subset S$ be a monomial ideal with $k > 1$ minimal generators. Then the following are equivalent:*

- (1) $L_I \cong \mathfrak{B}(k)$, i.e. I has the lcm-semilattice of a complete intersection.
- (2) $\text{pdim}_S S/I = k$.
- (3) $\text{spdim}_S S/I = k$.

Moreover, if $\text{pdim}_S S/I = k - 1$ then S/I and I satisfy Conjecture 2.5 (Stanley conjecture).

Proof. If $L_I \cong \mathfrak{B}(k)$ then $\text{spdim}_S S/I = k$ by Proposition 5.2. Moreover, in this situation $\text{pdim}_S S/I = k$ because this is the projective dimension of a k -generated complete intersection. So we need to show that $L_I \not\cong \mathfrak{B}(k)$ implies that $\text{spdim}_S S/I \leq k - 1$ and $\text{pdim}_S S/I \leq k - 1$.

By Remark 5.1 there exists a surjective join-preserving map $\mathfrak{B}(k) \longrightarrow L_I$. As $L_I \not\cong \mathfrak{B}(k)$ this map is not injective, so by Lemma 4.4 it factors through $\mathfrak{B}(k)/\sim_a$ for some meet-irreducible element $a \in \mathfrak{B}(k)$. But the automorphism group of $\mathfrak{B}(k)$ acts transitively on the set of meet-irreducible elements, so $L := \mathfrak{B}(k)/\sim_a$ does not depend on a . Let J be a monomial ideal (in some polynomial ring S') whose lcm-semilattice equals L , which does exist by Corollary 3.6. Now Theorems 4.5 and 4.9 imply that it suffices to prove $\text{spdim}_{S'} S'/J \leq k - 1$ and $\text{pdim}_{S'} S'/J \leq k - 1$.

We claim that we can choose

$$J = (x_1^2, \dots, x_{k-1}^2, x_1 x_2 \cdots x_{k-1}) \subset \mathbb{K}[x_1, \dots, x_{k-1}].$$

To see this, let us identify each element of $\mathfrak{B}(k)$ by the set of atoms below it. Then—up to an automorphism—we have $a = \{1, \dots, k - 1\}$. The meet-irreducible elements of L are the $(k - 1)$ -subsets of $[k] := \{1, \dots, k\}$ other than a , and the $(k - 2)$ -subsets of a . We

can choose a map w as in Theorem 3.4: Set $w([k] \setminus \{i\}) = w(a \setminus \{i\}) = x_i$ and all other elements of L are mapped to 1. It is easy to see that this map satisfies the condition of Theorem 3.4 and that the generators of the corresponding ideal are as claimed. Here, x_i^2 corresponds to the atom $\{i\}$ for $1 \leq i \leq k-1$ and $x_1 x_2 \cdots x_{k-1}$ corresponds to $\{k\}$.

Now J is a monomial ideal in $k-1$ variables, so the claimed inequalities $\text{spdim}_{S'} S'/J \leq k-1$ and $\text{pdim}_{S'} S'/J \leq k-1$ hold trivially. Further, by [Her13, Theorem 27] it holds that $\text{sdepth} J > 0$ and thus $\text{spdim} I \leq \text{spdim} J \leq k-2$. \square

5.2. Deformations of monomial ideals. The notion of *deformation* of a monomial ideal was introduced by Bayer et al. [BPS98] and further developed in Miller et al. [MSY00]. In order to include the case of quotients I/J , we slightly extend the definition found in [MSY00].

Definition 5.4. (1) Let $\mathcal{M} = \{m_1, \dots, m_r\} \subset S$ be a set of monomials. For $1 \leq i \leq r$ let $a^i = (a_1^i, \dots, a_n^i) \in \mathbb{N}^n$ denote the exponent vector of m_i . A *deformation* of \mathcal{M} is a set of vectors $\varepsilon_i = (\varepsilon_1^i, \dots, \varepsilon_n^i) \in \mathbb{N}^n$ for $1 \leq i \leq r$ subject to the following conditions:

$$a_j^i > a_j^k \implies a_j^i + \varepsilon_j^i > a_j^k + \varepsilon_j^k \quad \text{and} \quad a_k^i = 0 \implies \varepsilon_k^i = 0. \quad (5.1)$$

(2) Let $J \subsetneq I \subset S$ be two monomial ideals with generating sets G_I and G_J . A *common deformation* of I and J is a deformation of the union $G_I \cup G_J$. We set $I_\varepsilon := (g \cdot \mathbf{x}^{\varepsilon_g} : g \in G_I)$ and $J_\varepsilon := (g \cdot \mathbf{x}^{\varepsilon_g} : g \in G_J)$ to be the ideals generated by the deformed generators.

Remark that the equation (5.1) implies that $J_\varepsilon \subsetneq I_\varepsilon$.

Proposition 5.5. *Let $J \subsetneq I \subset S$ be two monomial ideals and let $J_\varepsilon \subsetneq I_\varepsilon \subset S$ be a deformation of I and J . Then*

$$\text{sdepth}_S I/J \geq \text{sdepth}_S I_\varepsilon/J_\varepsilon$$

Proof. As noticed in [GPW99], the map sending each deformed monomial $g \cdot \mathbf{x}^{\varepsilon_g}$ to the corresponding original monomial g induces a surjective join-preserving map from the lcm-semilattice of the generators of I_ε and J_ε to the lcm-semilattice of the generators of I and J . \square

The most important deformations are the generic deformations. Let us recall the definition from [MSY00].

Definition 5.6. (1) A monomial $m \in S$ is said to *strictly divide* another monomial $m' \in S$ if $m \mid \frac{m'}{x_i}$ for each variable x_i dividing m' .

(2) A monomial ideal $I \subset S$ is called *generic* if for any two minimal generators m, m' of I having the same degree in some variable, there exists a third minimal generator m'' that strictly divides $\text{lcm}(m, m')$.

(3) A deformation of a monomial ideal I is called *generic* if the deformed ideal I_ε is generic.

Corollary 5.7. *If $I \subset S$ is a monomial ideal such that $\text{depth}_S S/I = \text{depth}_S S/I_\varepsilon$ for some generic deformation of I , then $\text{sdepth}_S S/I \geq \text{depth}_S S/I$ (i. e. Stanley conjecture holds for S/I).*

Proof. It was proven by Apel in [Ape03b] that $\text{sdepth}_S S/J \geq \text{depth}_S S/J$ for every generic monomial ideal J . So the claim follows from Proposition 5.5 by considering the generic deformation of I . \square

5.3. Colon ideals and associated primes. In this subsection we consider colon ideals with respect to monomials. Both results of this section can be proven directly in a similar way to Lemma 4.6. However, we include them because we would like to illustrate that they also follow from our main result. Moreover, our proof works uniformly for the depth and the Stanley depth. The first result generalizes both [Cim09, Theorem 1.1] and Rauf [Rau10, Corollary 1.3].

Proposition 5.8. *Let $J \subsetneq I \subset S$ be two monomial ideals and let $v \in S$ be a monomial. Then*

$$\text{spdim}_S I/J \geq \text{spdim}(I : v)/(J : v)$$

and the same holds for the projective dimension.

Proof. Let $L_\infty \subset S$ denote the set of all monomials in S , ordered by divisibility. Let further $L := L_{G(I) \cup G(J)}$ and $L' := L_{G(J)}$ where $G(I)$ and $G(J)$ are sets of generators of I and J . We consider the map

$$\phi : L \rightarrow L_\infty, \quad m \mapsto \frac{m \vee v}{v}.$$

It is easy to see that ϕ preserves the join. We will show that the image $\phi(L)$ (as a set of monomials) generates $I : v$ and similarly $\phi(L')$ generates $J : v$, so our claim follows from Theorem 4.5 (resp. Theorem 4.9).

By symmetry, we only consider L . It is clear that $\phi(L) \subseteq I : v$. For the other inclusion consider a monomial $m \in I : v$. Then there exists a generator g of I such that $g \mid vm$. Hence $g \vee v \mid vm$ and thus $\phi(g) \mid m$. So $I : v$ is contained in the ideal generated by the image of ϕ . \square

As a consequence, we get the well-known bound on the depth and Stanley depth in terms of the height of associated prime ideals, see [Her13, Theorem 9].

Corollary 5.9. *Let $I \subset S$ be a monomial ideal. If I has an associated prime $\mathfrak{p} \subset S$ of height p , then*

$$\begin{aligned} \text{spdim}_S S/I, \text{pdim}_S S/I &\geq p \\ \text{spdim}_S I &\geq \lfloor \frac{p}{2} \rfloor \\ \text{pdim}_S I &\geq p - 1 \end{aligned}$$

Proof. This follows from the foregoing proposition, given the known values of spdim and pdim for monomial prime ideals. \square

From the proof of Proposition 5.8 one can also extract the following lattice-theoretical statement, which might be of independent interest. As we do not use it, we omit the proof.

Proposition 5.10. *Let L be a finite atomistic semilattice and let $p \in \mathbb{N}$. The following are equivalent:*

- (1) *There exists a surjective join-preserving map $L \rightarrow \mathfrak{B}(p)$ onto the boolean semilattice on p generators.*

- (2) *There exists a monomial ideal I with $L \cong L_I$ and I has an associated prime of height p .*

5.4. Ideals generated in a single degree. In this subsection we show that every finite semilattice can be realized as lcm-semilattice of a monomial ideal, whose minimal generators all have the same degree.

Lemma 5.11. *Let L be a finite semilattice and let $A \subset L$ be an antichain, i.e. a set of pairwise incomparable elements. Then there exists a (finite) set of monomials $G \subset S$ such that $L \cong L_G$ and the monomials in L_G corresponding to the elements of A all have the same degree.*

Proof. First, choose a map $w_1 : \bar{L} \rightarrow S$ satisfying the conditions of Theorem 3.4, where S is some polynomial ring with sufficiently many variables. If the monomials m_a for $a \in A$ (as in (3.1)) have all the same degree, then we are already done. Otherwise, let $a_1, a_2, \dots, a_r \in A$ be the elements whose monomials have the maximum degree among the monomials corresponding to A . We modify our w_1 by setting

$$w_2(m) := \begin{cases} X_i w_1(m) & \text{if } m = a_i, \\ w_1(m) & \text{otherwise.} \end{cases}$$

where the X_i are new variables. As A is an antichain, it follows from (3.1) that the degree of the monomials corresponding to $a_1, a_2, \dots, a_r \in A$ under w_2 increases by $r - 1$, while the degree of all other monomials corresponding to A increases by r . Hence after iterating this procedure finitely many times, all monomials corresponding to A have the same degree. \square

Proposition 5.12. *Let $J \subsetneq I \subset S$ be two monomial ideals. Then one can find monomial ideals $J' \subsetneq I' \subset S'$ with $L_{G(I) \cup G(J)} \cong L_{G(I') \cup G(J')}$ where the isomorphism maps $L_{G(J)}$ to $L_{G(J')}$, such that one's choice of the following holds:*

- (1) *Either I' is generated in a single degree, or*
- (2) *J' is generated in a single degree.*

In particular, to prove the Stanley conjecture for I/J , one may always assume that one of the ideals is generated in a single degree.

Proof. The set of minimal generators of I forms an antichain in $L_{G(I) \cup G(J)}$. Applying the foregoing Lemma 5.11 to it yields ideals $J' \subset I'$ with the same lcm-semilattice, where I' is generated in a single degree. On the other hand, applying the lemma to the antichain formed by the minimal generators of J results in J' being generated in a single degree. \square

Remark 5.13. Note that our construction does in general not allow to assume that both ideals are generated in a single degree. However, if $S = I$ (or more generally if one ideal is principal), then this is possible. Moreover, the introduction of many new variables will almost surely cause I/J to be not Cohen-Macaulay. So this reduction is not compatible with the reduction to the Cohen-Macaulay case given in [HJZ10].

5.5. Further applications. From the fact that every finite atomistic semilattice is the image of some $\mathfrak{B}(k)$, we can derive the following “local” version of the Stanley conjecture.

Theorem 5.14. *The following are equivalent:*

- (1) *The Stanley conjecture holds for S/I for every monomial ideal I .*
- (2) *If the Stanley conjecture holds for S/I a monomial ideal I , then it also holds for every monomial localization of S/I .*

The same equivalence holds for monomial ideals I instead of quotients S/I , where one takes images of I under the map sending variables to 1 instead of localization.

Proof. It is known that the maximal ideal satisfies the Stanley conjecture. By Remark 5.1, together with Lemmata 4.7 and 4.2, every quotient S/I can be seen as a localization of a quotient S'/I' , where I' has the same lcm-semilattice as the maximal ideal. \square

Remark 5.15. In order to prove the Stanley conjecture for a module of the form S/I , it is enough to consider the case when S/I is Cohen-Macaulay (see Herzog et al. [HJY08], [HJZ10]). But if S/I is Cohen-Macaulay and satisfies the Stanley conjecture, the so does every monomial localization of it. This follows via polarization from the fact that every link of a partitionable simplicial complex is itself partitionable.

So one might easily arrive at the false conclusion that this together with Theorem 5.14 implies the Stanley conjecture. The reason why this does not work is the following: Even though Theorem 5.14 allows us to assume that S/I is the localization of another quotient S'/I' satisfying the conjecture, S'/I' might fail to be Cohen-Macaulay.

Finally, let us point out some more results that also follow from our main result. The following result was originally proven in Ishaq [Ish12]. See also Seyed Fakhari [Fak13] for a different proof.

Proposition 5.16. *Let $J \subsetneq I \subset S$ be two monomial ideals. Then*

$$\text{sdepth}_S \sqrt{I}/\sqrt{J} \geq \text{sdepth}_S I/J.$$

Proof. Let $G(I)$ and $G(J)$ be minimal generating sets of I and J . For a monomial m we write \sqrt{m} for product of the variables dividing m . Define $G'(I)$ as $\{\sqrt{m} : m \in G(I)\}$ and $G'(J)$ similarly. Then $G'(I)$ and $G'(J)$ generate \sqrt{I} and \sqrt{J} . Moreover, the map $m \mapsto \sqrt{m}$ gives rise to a surjective join-preserving map $L_{G(I) \cup G(J)} \rightarrow L_{G'(I) \cup G'(J)}$. So the claim follows from Theorem 4.5. \square

Remark 5.17. There are several operations known on monomial ideals which do not change the lcm-lattice. In all these cases, the effect on depth and Stanley depth can be deduced from Theorem 4.5 and Theorem 4.9. We give references to several articles where these operations were previously studied:

- Polarization. However, we used this result (cf. [IKMF14]) in our proof of Theorem 4.5.
- Multiplication of an ideal by a monomial [Cim09, Theorem 1.4].
- The quotient modulo a non-zerodivisor monomial [Rau07, Theorem 1.1], and the extension of the polynomial ring by new variables [IMF14, Prop. 5.1].
- The constructions in Propositions 5.1 and 5.2 of [IKMF14], which themselves are extensions of other results (Ishaq and Qureshi [IQ11, Lemma 2.1], [Cim08, Lemma 1.1], [She09, Lemma 2.3]).
- Corollary 3.3. in Yanagawa [Yan12] and the properties treated in Anwar and Popescu [Pop14].

6. A STANLEY CONJECTURE FOR (SEMI-)LATTICES?

In view of our main result, the Stanley projective dimension of an ideal depends only on its lcm-semilattice. Hence one can interpret this number as a combinatorial invariant of the semilattice itself. This leads to a purely lattice theoretical formulation of the Stanley conjecture. Let us make this precise.

Definition 6.1. Let L be a finite semilattice. Choose an ideal $I \subset S = \mathbb{K}[X_1, \dots, X_n]$ such that $L_I \cong L$. We define

- (1) $\text{pdim}_1 L := \text{pdim } I$,
- (2) $\text{pdim}_2 L := \text{pdim } S/I$,
- (3) $\text{spdim}_1 L := \text{spdim } I$ and
- (4) $\text{spdim}_2 L := \text{spdim } S/I$.

Note that it trivially holds that $\text{pdim}_1 L = \text{pdim}_2 L - 1$ and that these invariants may depend on the underlying field \mathbb{K} .

The Stanley conjecture for ideals and for quotients (Conjecture 2.5), as well as the conjecture that $\text{sdepth } I \geq \text{sdepth } S/I + 1$ [Her13, Conjecture 64] can now be formulated in the following terms:

Conjecture 6.2. *For all finite semilattices L , the following inequalities hold:*

- (1) $\text{spdim}_1 L \leq \text{pdim}_1 L$,
- (2) $\text{spdim}_2 L \leq \text{pdim}_2 L$, and
- (3) $\text{spdim}_1 L \leq \text{spdim}_2 L - 1$.

A natural question is how these new invariants relate to the usual invariants of (semi-)lattices. Proposition 5.2 can be interpreted in this way, where the number of generators k corresponds to the width of the subposet of join-irreducible elements of L . Another step in this direction was taken by the second author together with S.A. Seyed Fakhari in [KSF14], where we show that $\text{spdim}_1 L$ and $\text{spdim}_2 L$ are bounded above by the length and by the order dimension of L .

One problem with our Definition 6.1 is that one has to choose an ideal I . While this is not difficult in practice due to Theorem 3.4, it would be conceptually nicer to be able to avoid this choice. Therefore we ask:

Question 6.3. *Is there a purely lattice theoretic description of $\text{spdim}_1 L$ and $\text{spdim}_2 L$?*

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